Formal Definition of a Group

A group is a set G of elements and a binary operation $\bullet$ which satisfies the following axioms:

1. For any two elements a, b in G, $a \bullet b$ is also in G (closure)

2. For any three elements a, b, c in G $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ (associativity)

3. There is an element e in G such that for any a in G $a\cdot e = a = e\cdot a$ (identity)

4. For any a in G, there is an element a' in G such that $a\cdot a' = e = a'\cdot e$ (inverse)
   We write $a' = a^{-1}$.

A binary operation is a way to take 2 elements of a set and make a new one.

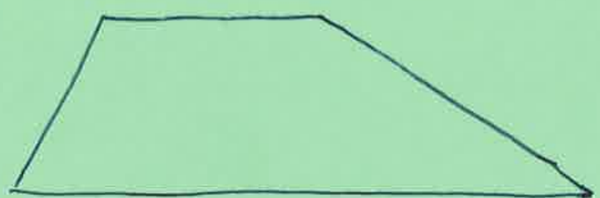
Ex: Give an example of a shape with full symmetry group:

(a) The trivial group $\{e\}$ (i.e., no symmetries

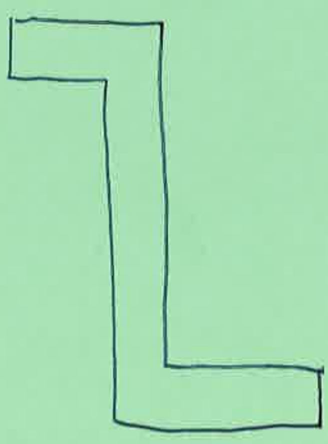
(b) $\mathbb{Z}_2$ (c) $\mathbb{Z}_3$ (d) $\mathbb{Z}_6$

Sol:

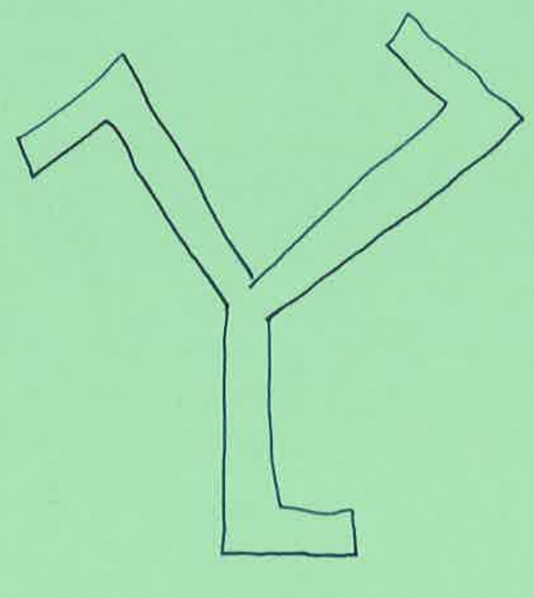
(a)



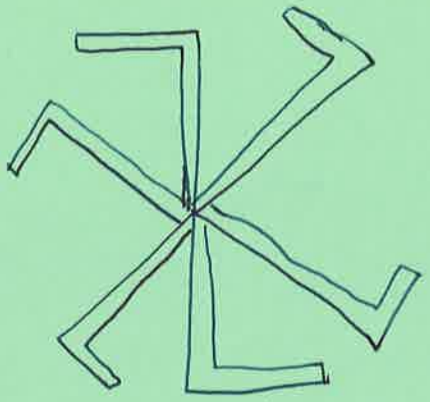
(b)



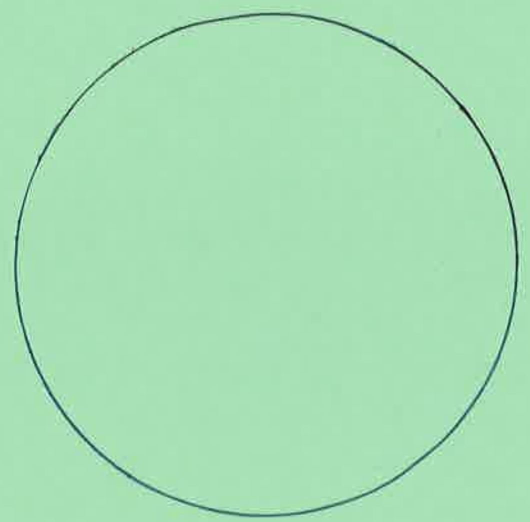
(c)



(d)



What is the full symmetry group of the circle?



- Rotations by <u>any</u> angle θ
- Reflections across <u>any</u> line through the center of the circle

So, the symmetry group of the circle is infinite!

Let's call counterclockwise rotation by θ : R_θ and reflection across the horizontal line : S

Notice that any other reflection is a combination or S and R_θ for some θ.

This is similar to the dihedral groups, D_n, except that there isn't a "smallest rotation" this time!

Question: ⓐ What is R_θ^{-1}?

ⓑ Do you think we have a relation

$$SR_\theta = R_\theta^{-1} S$$

like before?

Answer: ⓐ $R_\theta^{-1} = \underline{R_{-\theta}} = R_{2\pi-\theta}$

this one is better, I think.

ⓑ It actually is true, but really difficult to precisely show at the moment.

The symmetry group of the circle is an <u>Orthogonal Group</u>, specifically, $O(2)$.

The physical symmetry group (gets rid of reflections) is a Special Orthogonal Group, specifically, $SO(2)$.

To understand these groups, we need matrices. $O(2)$ and $SO(2)$ (and more generally $O(n)$ and $SO(n)$) are matrix Lie groups.

<u>What are matrices?</u>

They are certain kinds of functions which take in a vector and output a new vector. They also satisfy

$$f(c\vec{v}+k\vec{w}) = cf(\vec{v})+kf(\vec{w})$$

for any constants c, k and any vectors $\vec{v}, \vec{w}$ in their domain. We write matrices as a box of numbers: